$g(\cdot)$

10.5 Function of Continuous Random Variables: SISO

Reconsider the derived random variable $Y = g(X)$.

$$X \rightarrow \boxed{g(\cdot)} \rightarrow Y = g(X)$$

Ex. $Y = X^2$ $\quad g(\cdot) = (\cdot)^2$

$Y = 4|X - 1.5|$

$g(\cdot) = 4|\cdot - 1.5|$

Recall that we can find $\mathbb{E}Y$ easily by (22):

$$\mathbb{E}Y = \mathbb{E}\left[g(X)\right] = \int_{\mathbb{R}} g(x) f_X(x) dx.$$

However, there are cases when we have to evaluate probability directly involving the random variable Y or find $f_Y(y)$ directly.

Recall that for discrete random variables, it is easy to find $p_Y(y)$ by adding all $p_X(x)$ over all x such that $g(x) = y$:

$$p_Y(y) = \sum_{x:g(x)=y} p_X(x). \tag{23}$$

For continuous random variables, it turns out that we can't[45] simply integrate the pdf of X to get the pdf of Y.

10.61. For $Y = g(X)$, if you want to find $f_Y(y)$, the following ***two-step procedure*** will always work and is easy to remember:

(a) Find the cdf $F_Y(y) = P[Y \le y]$.

(b) Compute the pdf from the cdf by "finding the derivative" $f_Y(y) = \frac{d}{dy} F_Y(y)$ (as described in 10.13).

Affine

10.62. ***Linear Transformation***: Suppose $Y = aX + b$. Then, the cdf of Y is given by

1 $$F_Y(y) = P[Y \le y] = P[aX + b \le y] = \begin{cases} P\left[X \le \frac{y-b}{a}\right], & a > 0, \\ P\left[X \ge \frac{y-b}{a}\right], & a < 0. \end{cases}$$

Now, by definition, we know that

$$P\left[X \le \frac{y-b}{a}\right] = F_X\left(\frac{y-b}{a}\right),$$

[45] When you applied Equation (23) to continuous random variables, what you would get is $0 = 0$, which is true but not interesting nor useful.

and

$$P\left[X \geq \frac{y-b}{a}\right] = P\left[X > \frac{y-b}{a}\right] + P\left[X = \frac{y-b}{a}\right]$$
$$= 1 - F_X\left(\frac{y-b}{a}\right) + P\left[X = \frac{y-b}{a}\right].$$

For continuous random variable, $P\left[X = \frac{y-b}{a}\right] = 0$. Hence,

$$\boxed{F_Y(y) = \begin{cases} F_X\left(\frac{y-b}{a}\right), & a > 0, \\ 1 - F_X\left(\frac{y-b}{a}\right), & a < 0. \end{cases}}$$

Finally, fundamental theorem of calculus and chain rule gives

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \begin{cases} \frac{1}{a}f_X\left(\frac{y-b}{a}\right), & a > 0, \\ -\frac{1}{a}f_X\left(\frac{y-b}{a}\right), & a < 0. \end{cases}$$

$Y = 3X + 1 \to f_Y(y) = \frac{1}{3} f_X\left(\frac{y-1}{3}\right)$

Note that we can further simplify the final formula by using the $|\cdot|$ function: $Y = aX + b$

$$\boxed{f_Y(y) = \frac{1}{|a|}f_X\left(\frac{y-b}{a}\right), \quad a \neq 0.} \tag{24}$$

$Y = -5X + 2$

$f_Y(y) = \left|-\frac{1}{5}\right| f_X\left(\frac{2-y}{5}\right)$

Graphically, to get the plots of f_Y, we compress f_X horizontally by a factor of a, scale it vertically by a factor of $1/|a|$, and shift it to the right by b.

Of course, if $a = 0$, then we get the uninteresting degenerated random variable $Y \equiv b$.

Example 10.63. Suppose $X \sim \mathcal{E}(\lambda)$. Let $Y = 5X$. Find $f_Y(y)$.

$a = 5$, $b = 0$

$F_Y(y) = P[Y \leq y] = P[5X \leq y]$
$= P\left[X \leq \frac{y}{5}\right] = F_X\left(\frac{y}{5}\right)$

$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$

We can use (24) to get $f_Y(y)$ directly.

chain rule

$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X\left(\frac{y}{5}\right) = \frac{1}{5} f_X\left(\frac{y}{5}\right) = \begin{cases} \frac{1}{5}\lambda e^{-\lambda \frac{y}{5}}, & \frac{y}{5} > 0 \\ 0, & \text{otherwise} \end{cases}$

$= \begin{cases} \frac{\lambda}{5} e^{-\frac{\lambda}{5}y}, & y > 0, \\ 0, & \text{otherwise.} \end{cases}$

10.64. Suppose $X \sim \mathcal{N}(m, \sigma^2)$ and $Y = aX + b$ for some constants a and b. Then, we can use (24) to show that $Y \sim \mathcal{N}(am + b, a^2\sigma^2)$.

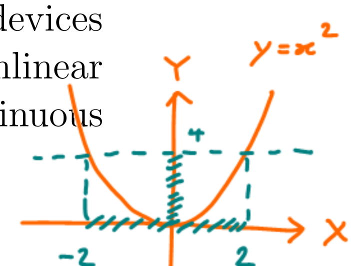
$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) = \frac{1}{\sqrt{2\pi}\,\sigma|a|} \exp\left(-\frac{1}{2\sigma^2}\left(\frac{y-b}{a} - m\right)^2\right)$$

$$\frac{y-b}{a} - m = \frac{y-(b+am)}{a}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

$$= \frac{1}{\sqrt{2\pi}(\sigma|a|)} \exp\left(-\frac{1}{2}\left(\frac{y-(am+b)}{\sigma|a|}\right)^2\right)$$

Example 10.65. Amplitude modulation in certain communication systems can be accomplished using various nonlinear devices such as a semiconductor diode. Suppose we model the nonlinear device by the function $Y = X^2$. If the input X is a continuous random variable, find the density of the output $Y = X^2$.



$$F_Y(y) = P[Y \le y] = P[X^2 \le y] = \begin{cases} 0, & y < 0 \\ \dots \end{cases}$$

$f_Y(y) = 0$ for $y < 0$

For $y > 0$, $F_Y(y) = P[X^2 \le y] = P[-\sqrt{y} \le X \le \sqrt{y}] = F_X(\sqrt{y}) - F_X(-\sqrt{y})$

$$f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) - \left(-\frac{1}{2\sqrt{y}}\right) f_X(-\sqrt{y})$$

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}}\left(f_X(\sqrt{y}) + f_X(-\sqrt{y})\right), & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Example 10.66. Suppose $X \sim \mathcal{E}(\lambda)$. Let $Y = \frac{1}{X^2}$. Find $f_Y(y)$.

$$F_Y(y) = P[Y \le y] = P\left[\frac{1}{X^2} \le y\right] = \begin{cases} 0, & y < 0 \\ P\left[X^2 \ge \frac{1}{y}\right], & y > 0 \end{cases}$$

$X \ge 0$, always

For $y > 0$,

$$F_Y(y) = P\left[X \ge \frac{1}{\sqrt{y}}\right] + \cancel{P\left[X \le -\frac{1}{\sqrt{y}}\right]} = 1 - F_X\left(\frac{1}{\sqrt{y}}\right)$$

$$\frac{d}{dy}\frac{1}{\sqrt{y}} = \frac{d}{dy} y^{-\frac{1}{2}} = -\frac{1}{2} y^{-\frac{3}{2}}$$

$$f_Y(y) = \frac{1}{2} y^{-\frac{3}{2}} f_X\left(\frac{1}{\sqrt{y}}\right) = \frac{1}{2} y^{-\frac{3}{2}} \lambda e^{-\lambda/\sqrt{y}}$$

$$f_Y(y) = \begin{cases} \frac{1}{2} y^{-\frac{3}{2}} \lambda e^{-\lambda/\sqrt{y}}, & y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

$Y = g(X)$, $g(x) = \frac{1}{x^2}$

① Find root(s):

$y = \frac{1}{x^2} \Rightarrow x = \pm\frac{1}{\sqrt{y}}$ for $y > 0$

No root when $y < 0$.

② Find slope(s)

$g(x) = \frac{1}{x^2} = x^{-2}$

$|g'(x)| = |-2x^{-3}| = \left|-\frac{2}{x^3}\right|$

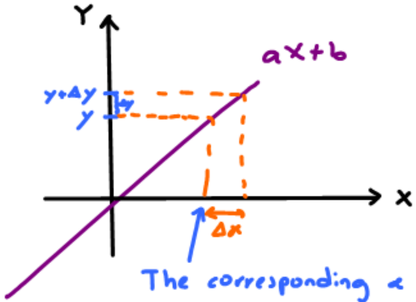
③ $$f_Y(y) = \begin{cases} \dfrac{f_X(1/\sqrt{y})}{\left|2/\left(\frac{1}{\sqrt{y}}\right)^3\right|} + \dfrac{\cancel{f_X(-1/\sqrt{y})}}{\left|2/\left(-\frac{1}{\sqrt{y}}\right)^3\right|}, & y > 0 \\ 0, & y < 0 \end{cases}$$

0

$$= \begin{cases} \frac{1}{2} y^{-3/2} f_X\left(\frac{1}{\sqrt{y}}\right), & y > 0 \\ 0, & y < 0 \end{cases}$$

same

$Y = aX + b$



The corresponding x for a y is $x = \frac{y-b}{a}$

$$P[y \le Y \le y+\Delta y] \approx P[x \le X \le x + \Delta x]$$

$$f_Y(y)|\Delta y| \approx f_X(x)|\Delta x|$$

$$f_Y(y) \approx f_X(x)\frac{|\Delta x|}{|\Delta y|} = f_X(x) \times \frac{1}{|\text{slope at } x|}$$

$$= f_X\left(\frac{y-b}{a}\right) \times \frac{1}{|a|}$$

Ex. $Y = X^2 \Rightarrow$ Here, $g(x) = x^2$.

① Find root(s) of $y = x^2$:

$\Rightarrow x = \pm\sqrt{y}$ ← two roots when $y > 0$.

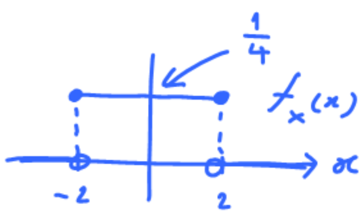
② Find slope(s):

$$g'(x) = \frac{d}{dx}x^2 = 2x$$

③ $$f_Y(y) = \begin{cases} \dfrac{f_X(\sqrt{y})}{|2\sqrt{y}|} + \dfrac{f_X(-\sqrt{y})}{|2(-\sqrt{y})|}, & y > 0, \\ 0, & y < 0. \end{cases}$$

Ex. $Y = e^X \Rightarrow$ Here, $g(x) = e^x$.

Assume $X \sim \mathcal{U}(-2, 2)$



(a) Suppose we want to find $f_Y(1)$.

① Find root: To get $Y = 1$, need $x = 0$.

② Find slope: $g'(x) = e^x \Rightarrow$ slope $= e^0 = 1$

③ $$f_Y(1) = \frac{f_X(0)}{1} = \frac{1/4}{1} = \frac{1}{4}$$

(b) Suppose we want to find $f_Y(-1)$. ① Find root: $-1 = e^x \Rightarrow$ No x satisfies this.

Therefore, $f_Y(-1) = 0$.

(c) Suppose we want to find $f_Y(100)$. ① Find root(s): $100 = e^x$

$\Rightarrow x = \ln(100) \approx 4.6$

② Find slope(s): $g(x) = e^x \Rightarrow g'(x) = e^x$.

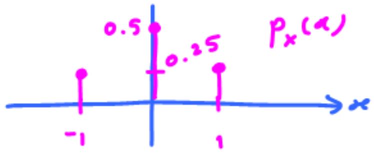
0 because $4.6 \notin (-2, 2)$

③ $$f_Y(100) = \frac{\cancel{f_X(4.6)}}{|e^{4.6}|} = 0$$

Goal: Given $f_X(x)$, suppose $Y = X^2$. Find $f_Y(y)$.

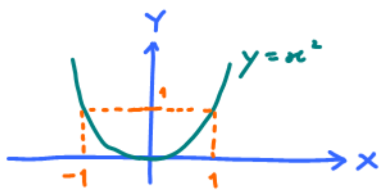
Discrete

$$p_X(x) = \begin{cases} 0.5, & x = 0, \\ 0.25, & x = 1, -1, \\ 0, & \text{otherwise.} \end{cases}$$



Let $Y = X^2$. Find $p_Y(1)$.

$\hookrightarrow P[Y = 1]$



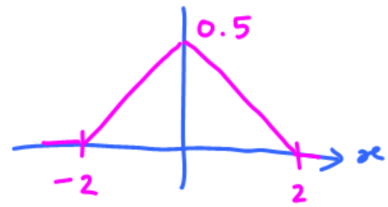
$$P[Y = 1] = P[X = 1] + P[X = -1]$$

$$= p_X(1) + p_X(-1)$$

$$= 0.25 + 0.25 = 0.5 = p_Y(1)$$

Continuous

$f_X(x) =$



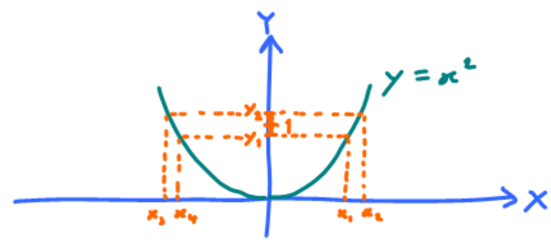
$$4 \times h \times \frac{1}{2} = 1$$

$$P[X = 0] = 0$$
$$P[X = -1] = 0$$
$$P[X = 1] = 0$$

Let $Y = X^2$. Find $P[Y = 1] = ?$

$$P[Y = 1] = P[X = 1] + P[X = -1]$$

$$0 = 0 + 0$$



Want to find $f_Y(1)$.

$$P[y_1 \le Y \le y_2] = P[x_1 \le X \le x_2] + P[x_3 \le X \le x_4]$$

$$\approx$$

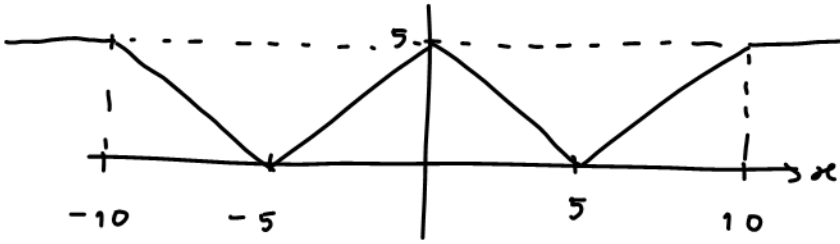
$$f_Y(1)|(y_2 - y_1)| \approx f_X(1)|(x_2 - x_1)| + f_X(-1)|(x_4 - x_3)|$$

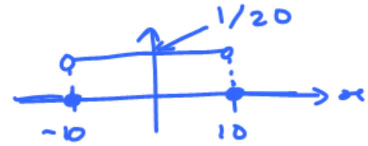
$$f_Y(1) \approx \frac{f_X(1)}{\left|\frac{y_2 - y_1}{x_2 - x_1}\right|} + \frac{f_X(-1)}{\left|\frac{y_2 - y_1}{x_4 - x_3}\right|}$$

$$= \frac{f_X(1)}{|\text{slope @ } 1|} + \frac{f_X(-1)}{|\text{slope @ } -1|}$$

Quiz #6. $X \sim \mathcal{U}(-10,10)$. Let $Y = g(X)$.

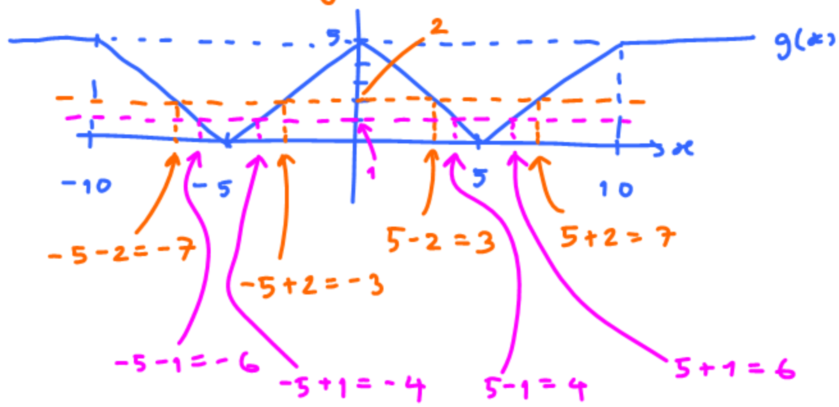
$$f_X(x) = \begin{cases} \frac{1}{20}, & -10 < x < 10, \\ 0, & \text{otherwise} \end{cases}$$





(a) Find $f_Y(2)$.

① Find root(s) of $g(x) = 2$. $\Rightarrow x = \pm 3, \pm 7$



② Find slopes: at $x = -3, 7$, $g'(x) = 1$.

at $x = 3, -7$, $g'(x) = -1$.

③ $$f_Y(2) = \frac{f_X(-7)}{|g'(-7)|} + \frac{f_X(-3)}{|g'(-3)|} + \frac{f_X(3)}{|g'(3)|} + \frac{f_X(7)}{|g'(7)|} = \frac{1/20}{|-1|} + \frac{1/20}{|1|} + \frac{1/20}{|-1|} + \frac{1/20}{|1|}$$

$$= \frac{1}{20} \times 4 = \frac{1}{5}.$$

(b) Find $f_Y(1)$.

Find root(s) of $g(x) = 1$ $\Rightarrow x = \pm 4, \pm 6$

$$f_Y(1) = \frac{f_X(-6)}{|g'(-6)|} + \frac{f_X(-4)}{|g'(-4)|} + \frac{f_X(4)}{|g'(4)|} + \frac{f_X(6)}{|g'(6)|}$$

$$= \frac{1/20}{|-1|} + \frac{1/20}{|1|} + \frac{1/20}{|-1|} + \frac{1/20}{|1|} = \frac{1}{20} \times 4 = \frac{1}{5}$$

(c) Repeat (a) and (b) but use $X \sim \mathcal{E}(1)$. $f_X(x) = \begin{cases} e^{-x}, & x > 0, \\ 0, & \text{otherwise} \end{cases}$

(a) $$f_Y(2) = \frac{\cancel{f_X(-7)}^{\,0}}{|g'(-7)|} + \frac{\cancel{f_X(-3)}^{\,0}}{|g'(-3)|} + \frac{f_X(3)}{|g'(3)|} + \frac{f_X(7)}{|g'(7)|} = \frac{e^{-3}}{1} + \frac{e^{-7}}{1} \approx 0.0507$$

(b) $$f_Y(1) = \frac{\cancel{f_X(-6)}^{\,0}}{|g'(-6)|} + \frac{\cancel{f_X(-4)}^{\,0}}{|g'(-4)|} + \frac{f_X(4)}{|g'(4)|} + \frac{f_X(6)}{|g'(6)|} = \frac{e^{-4}}{1} + \frac{e^{-6}}{1} \approx 0.0208$$

Exercise 10.67 (F2011)**.** Suppose X is uniformly distributed on the interval $(1,2)$. ($X \sim \mathcal{U}(1,2)$.) Let $Y = \frac{1}{X^2}$.

(a) Find $f_Y(y)$.

(b) Find $\mathbb{E}Y$.

Exercise 10.68 (F2011)**.** Consider the function

$$g(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

Suppose $Y = g(X)$, where $X \sim \mathcal{U}(-2,2)$.

Remark: The function g operates like a **full-wave rectifier** in that if a positive input voltage X is applied, the output is $Y = X$, while if a negative input voltage X is applied, the output is $Y = -X$.

(a) Find $\mathbb{E}Y$.

(b) Plot the cdf of Y.

(c) Find the pdf of Y

	Discrete	Continuous
$P[X \in B] =$	$\sum\limits_{x \in B} p_X(x)$	$\int\limits_B f_X(x)dx$
$P[X = x] =$	$p_X(x) = F(x) - F(x^-)$	0
Interval prob.	$P^X((a,b]) = F(b) - F(a)$ $P^X([a,b]) = F(b) - F(a^-)$ $P^X([a,b)) = F(b^-) - F(a^-)$ $P^X((a,b)) = F(b^-) - F(a)$	$P^X((a,b]) = P^X([a,b])$ $= P^X([a,b)) = P^X((a,b))$ $= \int\limits_a^b f_X(x)dx = F(b) - F(a)$
$\mathbb{E}X =$	$\sum\limits_x x p_X(x)$	$\int\limits_{-\infty}^{+\infty} x f_X(x)dx$
For $Y = g(X)$,	$p_Y(y) = \sum\limits_{x:\, g(x)=y} p_X(x)$	$f_Y(y) = \frac{d}{dy} P[g(X) \le y]$. Alternatively, $f_Y(y) = \sum\limits_k \frac{f_X(x_k)}{\lvert g'(x_k)\rvert}$, x_k are the real-valued roots of the equation $y = g(x)$.
For $Y = g(X)$, $P[Y \in B] =$	$\sum\limits_{x: g(x) \in B} p_X(x)$	$\int\limits_{\{x: g(x) \in B\}} f_X(x)dx$
$\mathbb{E}[g(X)] =$	$\sum\limits_x g(x) p_X(x)$	$\int\limits_{-\infty}^{+\infty} g(x) f_X(x)dx$
$\mathbb{E}[X^2] =$	$\sum\limits_x x^2 p_X(x)$	$\int\limits_{-\infty}^{+\infty} x^2 f_X(x)dx$
$\text{Var}\, X =$	$\sum\limits_x (x - \mathbb{E}X)^2 p_X(x)$	$\int\limits_{-\infty}^{+\infty} (x - \mathbb{E}X)^2 f_X(x)dx$

Table 5: Important Formulas for Discrete and Continuous Random Variables